COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **14** (3) (1994) 247–262

MINIMALLY 3-CONNECTED ISOTROPIC SYSTEMS

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Received July 7, 1990
Revised July 14, 1992

Isotropic systems are structures which unify some properties of 4-regular graphs and self-dual properties of binary matroids, such as connectivity and minors. In this paper, we find the minimally 3-connected isotropic systems. This result implies the binary part Tutte's wheels and whirls theorem.

1. Introduction

Isotropic systems are introduced by André Bouchet in [2]–[6] to unify some properties of 4-regular graphs and self-dual properties of binary matroids.

An isotropic system can be associated to a binary matroid (actually, to a pair of dual binary matroids) and also to a 4-regular graph, but there are isotropic systems that cannot be obtained in this way.

The connectivity of a matroid in Tutte's sense is self-dual. A. Bouchet defines in [6] the notion of k-connectivity for isotropic systems in such a way that an isotropic system S derived from a binary matroid is k-connected if and only if the matroid is k-connected.

Any isotropic system is associated with some simple graph, called *fundamental graph*, and the former is 3-connected if and only if the latter is prime, as defined by W. Cunningham [8].

A theorem of Tutte [9] says that a 3-connected matroid M has a 3-connected simple minor, except if M is either a wheel or a whirl. We prove that a similar result holds for 3-connected isotropic systems.

In Section 2, we define isotropic systems and recall basic properties. In Section 3, we give some technical lemmas about 3-connectivity. Section 4 contains the main result. This result is similar to Tutte's theorem about minor-minimal 3-connected matroids and, in fact, implies the "binary part" of this theorem. Moreover, the main result implies a strengthening of a theorem of A. Bouchet about reduction of prime graphs. These two applications are given in Section 5.

AMS subject classification code (1991): 05 B 35, 05 C 40

2. Notation and basic properties

Let A be a (finite) set. We denote by $|A|$ the number of elements of A. We will denote by a the singleton $\{a\}$.

For two sets A and B, we will denote by $A \triangle B$ the set $(A \backslash B) \cup (B \backslash A)$. For a set V, 2^V is the set of the subsets of V; 2^V is a linear space over GF_2 under the operation $\triangle$.

Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. For any vertex $v \in V$, the neighbourhood of v is the set of vertices $n(v)=\{u \in V : uv \in E\}$. Notice that $v \notin n(v)$. The mapping $v \rightarrow n(v)$ is the neighbourhood function of G. For any vertex v, $G \backslash v$ is the subgraph induced by $V \backslash v$.

2.1 Isotropic systems

Let us fix a 2-dimensional linear space K over GF_2. The space K contains $2^2=4$ elements. We set $K=\{0,x,y,z\}$. One easily verifies that $x+x=y+y=z+z=x+y+z=0$. For any a, $b \in K$, let $\langle a,b\rangle=1$ if $0 \neq a \neq b \neq 0$, $\langle a,b\rangle=0$ otherwise. One may verify that the mapping $(a,b) \rightarrow \langle a,b\rangle$ is bilinear, symmetric and nondegenerate. Let V be a finite set. The preceding linear form can be extended to K^V by setting $\langle X,Y\rangle=\sum(\langle X(v),Y(v)\rangle : v \in V) \pmod 2$ for any $X,Y \in K^V$. Notice that, for any $X \in K^V$, we have $\langle X,X\rangle=\sum(\langle X(v),X(v)\rangle : v \in V)=0$. We will denote by $L^\perp$ the orthogonal complement of a subspace L.

Let $X \in K^V$. The support of X is the set $\{v \in V : X(v) \neq 0\}$, and we denote it by $\sigma(X)$. For a subset P of V and a vector X, we denote by $X \cdot P$ the vector defined by $X \cdot P(v) = X(v)$ if $v \in P$, otherwise $X \cdot P(v) = 0$. For a vector $X \in K^V$, we let $\hat{X}=\{X \cdot P : P \subseteq V\}$. Clearly, for any subsets P and Q of V, we have $X \cdot P + X \cdot Q = X \cdot (P \triangle Q)$. Hence, $\hat{X}$ is a subspace of K^V. Two vectors X and Y of K^V are *supplementary* if, for any $v \in V$, we have $0 \neq X(v) \neq Y(v) \neq 0$. If X and Y are supplementary then $\sigma(X)=\sigma(Y)=V$ and $\hat{X} \cap \hat{Y}=\{0\}$. Hence, we have $\hat{X} \oplus \hat{Y}=K^V$.

A subspace L of K^V is said to be *totally isotropic* if $\langle A,B\rangle=0$ for any $A,B \in L$. If L is a totally isotropic subspace of K^V, basic results in linear algebra imply that $\dim(L) \leq \dim(K^V)/2=|V|$. An *isotropic system* is a pair $S=(L,V)$ where V is a finite set and L is a totally isotropic subspace of K^V such that $\dim(L)=|V|$. If $S=(L,V)$ is an isotropic system, then $L=L^\perp$.

Example. Suppose that M is a binary matroid on V. Let $\mathscr{C} \subseteq 2^V$ be the linear subspace of 2^V spanned by the circuits of M, and let $\mathscr{C}^*$ be the linear subspace spanned by the cocircuits of M. Recall that $\mathscr{C}^*$ is the orthogonal complement of $\mathscr{C}$ for the bilinear form defined on 2^V by $(A,B) \rightarrow |A \cap B| \pmod 2$.

Let X and Y be two supplementary vectors of K^V. Let $L=\{C \cdot X + C^* \cdot Y : C \in \mathscr{C}, C^* \in \mathscr{C}^*\}$. Then $S=(L,V)$ is an isotropic system.

Remark. One can derive several isotropic systems from a binary matroid, by arbitrarily choosing the vectors X and Y. But these isotropic systems are pairwise isomorphic. We will say "the isotropic system S derived from M", despite the fact that S is not completely defined by M.

2.2 Fundamental graphs

Let $G=(V,E)$ be a simple graph with neighbourhood function n. Consider two supplementary vectors A and B of K^V. For any $v \in V$, let T_V be the vector $A\cdot v+B\cdot n(v)$. One easily verifies that the linear subspace L spanned by $(T_v : v \in V)$ defines an isotropic system $S=(L,V)$. We say that G is a *fundamental graph* for this isotropic system and that (G,A,B) is a *graphic presentation* of the system S.

In [5], A. Bouchet proves that, for any isotropic system $S=(L,V)$, there exists some basis $(T_v : v \in V)$ of L with the following properties:

(i) $\forall u,v,w \in V, \quad v \neq u \neq w, \quad \langle T_v(u), T_w(u)\rangle = 0,$

(ii) $\forall u,v \in V, \quad u \neq v, \quad 0 \neq T_v(v) \neq T_u(v).$

Such a basis will be called a *fundamental basis*. It allows to define a graphic presentation (G,A,B) of S follows:

(iii) for any $v \in V$, $B(v)=T_v(v)$;

(iv) for any $v \in V$, if there exists $u \in V$ such that $T_u(v) \neq 0$, then $A(v)=T_u(v)$, otherwise $A(v)$ is any-null value distinct from $B(v)$;

(v) uv is an edge of G if and only if $u \neq v$ and $T_u(v) \neq 0$ (which is equivalent to $T_v(u) \neq 0$).

Let $G=(V,E)$ be a simple graph with neighbourhood function n. The *local complementation of* G *at a vertex* v involves replacing the subgraph induced by $n(v)$ by the complementary subgraph (an example is given in section 2.5). We denote this simple graph by G^*v. Suppose that (G,A,B) is a graphic presentation of an isotropic system S. Let $A'=A+B\cdot v$ and $B'=B+A\cdot n(v)$. Then, as proved in [5], $(G^*v,\ A',\ B')$ is another graphic presentation of S. Moreover, two simple graphs are fundamental graphs of the same isotropic system if and only if there exists a sequence of local complementations that transforms the first graph into the second one.

Example. Suppose that M is a binary matroid on V. Let B be a base of M. The fundamental graph F associated with a base B of M is a bipartite graph defined as follows:

— the two chromatic classes are B and $V\backslash B$;

— for $v \in B$ and $v' \in V\backslash B$, vv' is an edge of G if and only if v belongs to the single circuit of M in $B \cup v'$.

One can verify that F is a fundamental graph of the isotropic system S derived from the matroid M. Conversely, as proved in [4], an isotropic system S such that there exists a bipartite fundamental graph F is derived from a binary matroid. Such an isotropic system is called *bipartite*.

Remark. Notice that not every fundamental graph of a bipartite isotropic system is bipartite. Indeed, by locally complementing a bipartite graph at a vertex of degree at least two, we get a non-bipartite graph.

2.3 Connectivity

Let $S=(L,V)$ be an isotropic system. For any subset A of V, we let $L\times A=\{X\in L:\sigma(X)\subseteq A\}$. Let us identify K^A with the subspace $\{X\in K^V:\sigma(X)\subseteq A\}$. So we have $L\times A=L\cap K^A$, Clearly $L\times A$ is a subspace of K^V. We let $\chi(A)=|A|-\dim(L\times A)$, and we call χ the *connectivity function* of A.

Proposition 2.3.1. [6] *The connectivity function has the following properties:*

(i) $0\leq\chi(A)\leq|A|$;
(ii) $\chi(A)=\chi(V\backslash A)$;
(iii) $\chi(A)+\chi(B)\geq\chi(A\cup B)+\chi(A\cap B)$ *with equality if and only if* $L\times A+L\times B=L\times(A\cup B)$.

For completeness, we give a new proof of this proposition:
Proof.

(i) Notice that $L\times A$ is an isotropic subspace of K^A. Hence $\dim(L\times A)\leq\dim(K^A)/2=|A|$.

(ii) Let π be the canonical projection of K^V onto K^A and let $L\cdot A=\pi(L)$. Since $\mathrm{Ker}(\pi)\cap L=L\times(V\backslash A)$, we have

$$\dim(L\cdot A)+\dim(L\times(V\backslash A))=\dim(L)=|V|.$$

This implies $\chi(V\backslash A)=\dim(L\cdot A)-|A|$.

Let us denote by L_A the orthogonal complement of $L{\cdot}A$ in K^A. In other words, $L_A=\{X\mid\sigma(X)\subseteq A,\langle X,Y'\rangle=0\text{ for any }Y'\in L\cdot A\}$. Let $X\in L\times A$ and $Y'\in L\cdot A$. There exists $Y\in L$ such that $Y'=\pi(Y)$. Thus, we have $\langle X,Y'\rangle=\langle X,Y\rangle=0$. This implies that $L{\times}A\subseteq L_A$. Conversely, let $X\in L_A$ and $Y\in L$. Then $\pi(Y)\in L{\cdot}A$ and so $0=\langle X,\pi(Y)\rangle=\langle X,Y\rangle$. This implies that $X\in L^\perp$. Hence, as $\sigma(X)\subseteq A$ and $L=L^\perp$, it follows that $X\in L\times A$. Hence $L\times A=L_A$. Therefore, $\dim(L\times A)+\dim(L\cdot A)=\dim(K^A)=2|A|$. It follows:

$$\begin{aligned}\chi(A)&=|A|-\dim(L\times A)=|A|-(2|A|-\dim(L\cdot A))\\&=\dim(L\cdot A)-|A|=\chi(V\backslash A).\end{aligned}$$

(iii) Obviously, $L\times(A\cap B)=L\times A\cap L\times B$ and $L\times A+L\times B\subseteq L\times(A\cup B)$. Therefore, $\dim(L\times A)+\dim(L\times B)\leq\dim(L\times(A\cup B))+\dim(L\times(A\cap B))$. Since $|A|+|B|=|A\cup B|+|A\cap B|$, we have $\chi(A)+\chi(B)\geq\chi(A\cup B)+\chi(A\cap B)$, with equality if and only if $L\times A+L\times B=L\times(A\cup B)$. ∎

Definitions. Suppose that $S=(L,V)$ is an isotropic system, with connectivity function χ. For any integer k and any subset A of V, we say that A is a *k-separation* if and only if:

(i) $|A|\geq k$ and $|V\backslash A|\geq k$;
(ii) $\chi(A)<k$.

An isotropic system (L,V) is *k-connected* if and only if there is no k'-separation, with $k'<k$. An element v is *singular* if and only if $\chi(v)=0$. We easily verify that v is singular if and only if v is the support of a vector of L. Two non-singular

elements v and w are *twins* if and only if $\{v,w\}$ is the support of a vector of L. A *triangle* is a vector of L whose support is a 3-element set.

Suppose that (G,A,B) is a graphic presentation of (L,V), with fundamental basis $(T_v)_{v\in V}$. Denote by n the neighbourhood function of G. If a vector T of K^V belongs to L, there exists a subset V' of V such that $T=\sum(T_v : v\in V')$. As $v\in V'$ implies $T(v)\neq 0$ (actually $T(v)=B(v)$ or $T(v)=A(v)+B(v)$), V' must be included in the support of T. Using this remark, one easily verifies that:

(i) an element $v\in V$ is singular if and only if v is an isolated vertex of G;

(ii) two elements v and w are twins of L if and only if either v and w are twins in G (i.e, $n(v)-w=n(w)-v$) or vw is a pendent edge.

Example. Suppose that M is a binary matroid on a set V. Let M^* be the dual matroid of M. Denote by ϱ and ϱ^* the rank functions of M and M^*. The connectivity function of M is defined by $\chi(A)=\varrho(A)+\varrho^*(A)-|A|=\varrho(A)+\varrho(V\backslash A)-\varrho(V)$, for all $A\subseteq V$. It is proved in [5] that the connectivity function of M coincides with the connectivity function of the isotropic system S derived from M. So M is 3-connected if and only if S is 3-connected.

2.4 Minors

Let $S=(L,V)$ be an isotropic system. Let $v\in V$. For any non-null element t of K, we let $L(v,t)=\{X\in L : X(v)=t \text{ or } X(v)=0\}$. We define $L|_t^v$ as the projection onto K^{V-v} of $L(v,t)$. A. Bouchet proves in [2] that $(L|_t^v, V\backslash v)$ is an isotropic system. We call $(L|_t^v, V\backslash v)$ a *minor* of S and we denote it by $S|_t^v$. Minors of isotropic systems have the following graphic interpretation (see [5] for details). Suppose that (G,A,B) is a graphic presentation of L. Then $G\backslash v$ is a fundamental graph of $S|_{A(v)}^v$. Moreover, if G^*v is the graph obtained by a local complementation on v, then $G^*v\backslash v$ is a fundamental graph of $S|_{(A+B)(v)}^v$. Finally, if w is a neighbour of v in G, then $G^*vwv\backslash v$ is a fundamental graph of $S|_{B(v)}^v$.

Example. Let $S=(L,V)$ be the isotropic system derived from a binary matroid M. We use the notation of the example in Section 2.1. One easily verifies that the isotropic system derived from $M\backslash v$ (resp. M/v) is $S|_{Y(v)}^v$ (resp. $S|_{X(v)}^v$). The isotropic system $S|_{X(v)+Y(v)}^v$ is not binary in general.

2.5 Cyclic isotropic systems

An isotropic system $S=(L,V)$ is *cyclic* if there exists a graphic presentation (C_n,A,B) using the n-cycle C_n $(n\geq 5)$ as a fundamental graph. We shall assume $n\geq 5$ to avoid trivial special cases. The main result of this paper is that cyclic isotropic systems are "minimally" 3-connected. We now describe cyclic isotropic system more precisely.

Consider the isotropic system $S_n=(L,\{s_1,s_2,\ldots,s_n\})$, given by the fundamental basis $(T_i=T_{s_i}) : 1\leq i\leq n)$. Each vector T_i is a column of the following array:

	T_1	T_2	T_3	$\dots$	T_{n-1}	T_n
s_1	y	x	0	$\dots$	0	x
s_2	x	y	x	$\dots$	0	0
s_3	0	x	y	$\dots$	0	0
$\dots$						
s_{n-1}	0	0	0	$\dots$	y	x
s_n	x	0	0	$\dots$	x	y

The reader can easily verify that the associated fundamental graph is C_n with vertex sequence $s_1, s_2, \dots, s_n$. A local complementation at s_2 yields the graph $C_n^* s_2$. We notice that $C_n^* s_2 \backslash s_2$ is isomorphic to C_{n-1}. Hence one of the minors of S_n at s_2 is cyclic.

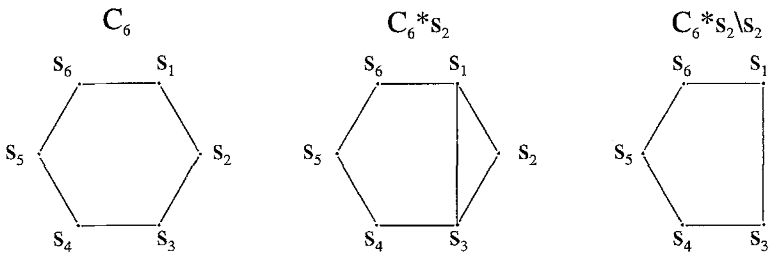

Fig. 1.

Isotropic systems associated to 4-regular graphs, as defined in [3], will not be considered in this paper.

3. Triangles and 3-connectivity

From now on, all isotropic systems contain at least four elements.

Lemma 3.1. *Let $S=(L,V)$ be an isotropic system, with connectivity function χ. For any subset A of V and any $v \in V \backslash A$, we have:*

(i) $\chi(A\cup v) \le \chi(A)$ *if and only if there exists a vector X in $L\times(A\cup v)$ such that $X(v)\neq 0$.*

(ii) $\chi(A\cup v)=\chi(A)-1$ *if and only if there exist two vectors X and Y in $L\times(A\cup v)$ such that $0\neq X(v)\neq Y(v)\neq 0$.*

Proof. Let $\pi : L\times(A\cup v)\to K$ be the linear mapping defined by $\pi(U)=U(v)$, for any vector U of $L\times(A\cup v)$. Since the kernel of π is $L\times A$, we have:

$$\dim(L\times(A\cup v)) = \dim(L\times A) + \dim(\mathrm{Im}(\pi)).$$

This implies:

$$\chi(A\cup v) = |A|+1-\dim(L\times(A\cup v)) = |A| - \dim(L\times A) - \dim(\mathrm{Im}(\pi)) + 1$$
$$\chi(A\cup v) = \chi(A) - \dim(\mathrm{Im}(\pi)) + 1.$$

(i) Hence, $\chi(A\cup v)\leq\chi(A)$ if and only if $\mathrm{Im}(\pi)$ is not the null subspace of K, i.e. if there exists a vector X in $L\times(A\cup v)$ such that $X(v)\neq 0$.

(ii) Furthermore, we have $\chi(A\cup v)=\chi(A)-1$ if and only if $\dim(\mathrm{Im}(\pi))=2$, i.e. if there exist two vectors X and Y in $L\times(A\cup v)$ such that $0\neq X(v)\neq Y(v)\neq 0$. ∎

Proposition 3.2. *Let $S=(L,V)$ be an isotropic system. Suppose that $v\in V$ is a non-singular element of L. Assume that χ (resp χ') is the connectivity function of S (resp. of $S|_t^v$, where t is a non-null element of k). Then, for any subset A of $V\backslash v$, we have:*

(i) $\chi(A)-1\leq\chi'(A)\leq\chi(A)$;

(ii) $\chi'(A)=\chi(A)$ *if and only if there exists a vector X in $L\times(A\cup v)$ such that* $X(v)=t$,

(iii) $\chi(A\cup v)\leq\chi'(A)\leq\chi(A\cup v)$.

Proof. Set $V'=V\backslash v$. Let $\pi: K^V\to K^{V'}$ be the canonical projection onto $K^{V'}$. For $X\in L$, $\pi(X)=0$ if and only if $X=0$, because v is non-singular. Hence, for any subspace L_1 of L, $\pi(L_1)$ is isomorphic to L_1 and so, $\dim(\pi(L_1))=\dim(L_1)$. Set $S'=(L',V')=S|_t^v$. For any subset A of V', we have $\chi'(A)=|A|-\dim(L'\times A)$. Set $L_A=L\times(A\cup v)\cap L(v,t)$. One can easily verify that $L'\times A=\pi(L_A)$. Hence $\dim(L'\times A)=\dim(L_A)$ because $L(v,t)$ is a subspace of L. Obviously, $L\times A\subseteq L_A\subseteq L\times(A\cup v)$. This implies the following inequalities:

$$\dim(L\times A)\leq\dim(L_A)\leq(L\times(A\cup v)),$$
$$|A|-\dim(L\times A)\geq|A|-\dim(L_A)\geq\dim(L\times(A\cup v)),$$
$$\chi(A)\geq\chi'(A)\geq\chi(A\cup v)-1.$$

Working with $V'\backslash A\,(=V\backslash(A\cup v))$ instead of A, we find:

$$\chi(V\backslash(A\cup v))\geq\chi'(V'\backslash A)\geq\chi(V\backslash A)-1.$$

By Proposition 2.3.1 (ii), this inequality implies:

$$\chi(A\cup v)\geq\chi'(A)\geq\chi(A)-1.$$

Moreover, $\chi'(A)=\chi(A)-1$ if and only if $L_A\neq L\times A$. If $X\in L_A\backslash L\times A$, then $X\in L\times(A\cup v)$ and $X(t)=t$. ∎

Lemma 3.3. *Suppose that T and T' are two triangles of a 3-connected isotropic system. Then one of the following properties holds:*

(i) *The supports are disjoints;*

(ii) *The supports contain a single common element. On this element, T and T' have the same value;*

(iii) *The supports have two common elements. On each common element, T and T' have different values;*

(iv) $T=T'$.

Proof. Since $\langle T,T'\rangle=0$, the number of the elements v that $\langle T(v),T'(v)\rangle\neq 0$ is even. If $T\neq T'$, the support of $T+T'$ must contain at least three elements because a 3-connected isotropic system contains neither singular elements nor twins. According to the number of elements in the intersection of the support of T and T', one of the properties above must hold.

Corollary 3.4.

(i) *No isotropic system on a 4-element set is 3-connected.*

(ii) *An isotropic system on a 5-element set is 3-connected if and only if it is cyclic.*

Comment. This corollary is equivalent to a result about prime graphs (cf. [2] and section 5 of this paper) which says that no graph on four vertices is prime and that all prime graphs on five vertices are locally equivalent to C_5.

Proof. Suppose that $S=(L,V)$ is a 3-connected isotropic system, with connectivity function χ.

(i) Suppose that $|V|=4$ and pick $a\in V$. We have $\chi(V\backslash a)=\chi(a)\leq 1$. Hence $L\times(V\backslash a)$ contains two independent vectors. If these two vectors are triangles, L is not 3-connected by Lemma 3.1. Otherwise S must contain singular elements or twins and is obviously not 3-connected.

(ii) Suppose now that $|V|=5$ and set $V=\{a,b,c,d,e\}$. If T is a 3-element subset of V then $\chi(T)=\chi(V\backslash T)=2$ and T is the support of a triangle. Hence, $\{a,b,c\}$, $\{b,c,d\}$, $\{c,d,e\}$, $\{d,e,a\}$ and $\{e,a,b\}$ are supports of triangles, which will be denoted by T_b, T_c, T_d, T_e, T_a. One may assume without loss of generality that $T_a(a)=T_b(b)=T_c(c)=T_d(d)=T_e(e)=y$. By Lemma 3.1, we have $0\neq T_a(b)=T_c(b)\neq T_b(b)$, $0\neq T_b(c)=T_c(c)\neq T_d(c),\ldots$ etc. One may suppose without loss of generality that all these values are equal to x. These five vectors are independent and so they constitute a basis of L. Hence C_5 is a fundamental graph of S.

	T_a	T_b	T_c	T_d	T_e
a	y	x	0	0	x
b	x	y	x	0	0
c	0	x	y	x	0
d	0	0	x	y	x
e	x	0	0	x	y

We have to prove that S is 3-connected. Suppose not and pick a 2-separation A. Then $V\backslash A$ or A contains two elements and S should contain twins, which is clearly impossible. ∎

Remark. Suppose that T is a triangle of an isotropic system $S=(L,V)$, with support $\{u,v,w\}$. The minor $S|^v_{T(v)}$ is not 3-connected, because u and w are twins in $L|^v_{T(v)}$.

Lemma 3.5. *Let (L,V) be a 3-connected isotropic system. Suppose that, for some $v\in V$, the minors $L|^v_x$ and $L|^v_y$ are not 3-connected. Then there exists a triangle T with $T(v)=x$ or $T(v)=y$.*

Proof. Assume that $U\subseteq V\backslash v$ (resp. $W\subseteq V\backslash v$) is a 2-separation of $S|^v_x$ (resp. of $S|^v_y$). Then $U'=V\backslash(U\cup v)$ and $W'=V\backslash(W\cup v)$ are also 2-separations in the corresponding minors.

Since S is 3-connected, $\chi(U)=2$ (where χ is the connectivity function in L). Then by Lemma 3.2, there exists a vector $T_x\in L$ such that $\sigma(T_x)\subseteq U+v\subseteq U\cup W+v$

and $T_x(v) = x$. Similarly $\chi(W) = 2$. Then, there exists $T_y \in L$ such that $T_y \subseteq U \cup W + v$ and $T_y(v) = y$. Hence, by Lemma 3.1:

$$\chi(U \cup W \cup v) = \chi(U \cup W) - 1,$$
$$\chi(U \cup W \cup v) + \chi(U \cap W) \leq \chi(U \cup W) + \chi(U \cap W) - 1 \leq \chi(U) + \chi(W) - 1 \leq 3.$$

This implies $|U \cap W| < 2$ or $|U' \cap W'| < 2$. By symmetry, $|U' \cap W| < 2$ or $|U \cap W'| < 2$. One of the four subsets U, U', W, W' must have two elements, say U. Then the 3-element set $U \cup v$ contains $\sigma(T_x)$. Since S is 3-connected, T_x is a triangle which satisfies the conclusion of the theorem. ∎

Lemma 3.5. *Let $S = (L, V)$ be a 3-connected isotropic system. Let T be a triangle, with $T(a) = T(c) = x$, $t(b) = Y$. Suppose that $L|_y^a$ and $L|_y^c$ are not 3-connected. Then there exist triangles T' and T'' such that:*
(i) $T'(a) = T''(c) = y$,
(ii) $T(b) = T''(b)$.

Proof. Let A be a 2-separation of $S|_y^a$. Let χ (resp. χ') be the connectivity function of S (resp. $S|_y^a$). Hence $\chi'(A) = 1$. Since S is a 3-connected, this implies that $\chi(A) = 2$ by Lemma 3.2. Then, by Lemma 3.2, there exists a vector T_a such that $T_a \in L \times (A \cup a)$ and $T(a) = y$. Since $A' = (V \backslash a) \backslash A$ is also a 2-separation in the minor $S|_{y'}^a$ there exists a vector T'_a such that $T'_a(a) = y$ and $T'_a \in L \times (A \cup a)$.

Similarly, there exists a 2-separation C in $L|_{y'}^c$, and two vectors T_c and T'_c of L such that:
— $T_c(c) = T'_c(c) = y$;
— $T_c \in L \times (C \cup v)$ and $T'_c \in L \times (C' \cup c)$;
— $\chi(C) = \chi(C \cup c) = 2$;
— $|C| \geq 2$, $|C'| \geq 2$.
By symmetry, we may assume that $b \in A \cap C$.

Claim 1. $a \in C'$, $c \in A'$

Suppose $a \in C$. Then $\langle T'_c, T \rangle = \langle T'_c(c), T(c) \rangle \neq 0$, a contradiction. Therefore $a \in C'$ and similarly $c \in A'$.
It follows that $\langle T'_c(a), T(a) \rangle = \langle T'_a(c), T(c) \rangle = 1$.

Claim 2. $|A \cap C| \leq 1$ *or* $|A' \cap C'| \leq 1$.

By submodularity, we get:

$$\chi(A \cup C) + \chi(A \cap C) \leq \chi(A) + \chi(C) \leq 4.$$

Since $T_a(a) \neq 0$ and $T_a \subseteq A \cup C \cup a$, we have $\chi(A \cup C \cup a) \leq \chi(A \cup C)$. Since $0 \neq T(c) \neq T_c(c) \neq 0$, $\sigma(T_c) \subseteq A \cup C \cup \{a, c\}$ and $\sigma(T) \subseteq A \cup C \cup \{a, c\}$, by Lemma 3.1, we have $\chi(A \cup C \cup \{a, c\}) \leq \chi(A \cup C) - 1$. It follows that $\chi(A \cup C \cup \{a, c\}) + \chi(A \cap C) \leq 3$.

Hence, since the complement of $A \cup C \cup \{a, c\}$ is $A' \cap C'$, we have $\chi(A' \cap C') \leq 1$ or $\chi(A \cap C) \leq 1$. This implies the claim by 3-connectivity.

Claim 3. *$A' \cap C = \emptyset$ or $(|A' \cap C| = |A \cap C'| = 1)$ or $A \cap C' = \emptyset$.*

By submodularity, we get:

$$\chi(A' \cup C) + \chi(A' \cap C) \leq \chi(A') + \chi(C).$$

Suppose that this inequality is actually an equality. Since $T_c \subseteq C \cup c \subseteq A' \cup C$, there exist two vectors T_1 and T_2 of L such that:

— $T_c = T_1 + T_2$,

— $\sigma(T_1) \subseteq C$, $\sigma(T_2) \subseteq (A' \cap C) \cup c$.

Thus $T_2(a) = T_2(b) = 0$, $T_1(c) = 0$, $T_2(c) = T_c(c) = y$. This implies that $\langle T, T_2 \rangle = \langle T(c), T_2(c) \rangle = \langle T(c), T_c(c) \rangle \neq 0$, a contradiction. Hence

$$\chi(A' \cup C) + \chi(A' \cap C) \leq 3.$$

Notice that $\langle T(a), T'_a(a) \rangle \neq 0$. Moreover, the supports of these two vectors are included in $A' \cup C \cup a$. This implies:

$$\chi(A \cap C') = \chi(A' \cup C \cup a) = \chi(A' \cup C) - 1,$$
$$\chi(A \cap C') + \chi(A' \cap C) \leq 2,$$

and the claim follows by 3-connectivity.

We end the proof by analizing the cases in Claim 2.

Case 1. $A \cap C = b$.

Neither $A' \cap C$ nor $A \cap C'$ is empty, for otherwise A or C would be a singleton. Hence, by Claim 3, T_a and T_c are triangles. Since $T_a \cap T_c = b$, $T_a(b) = T_c(b)$ by Lemma 3.1, we may set $T_1 = T_a$ and $T_2 = T_b$.

We may suppose from now on that $|A \cap C| > 1$ (and so $\chi(A \cap C) \geq 2$).

Case 2. $A' \cap C' = \emptyset$.

Neither $A' \cap C$ nor $A \cap C'$ is empty, for otherwise A' or C' would be a singleton. Then Claim 3 implies that $T_1 = T'_a$ and $T_2 = T'_b$ are triangles. These triangles satisfy the conclusion of the theorem.

Case 3. $A' \cap C'$ contains a single element, which will be denoted f.

By submodularity,

$$\chi(A \cup C) + \chi(A \cap C) \leq \chi(A) + \chi(C) \leq 4.$$

This implies:

$$\chi(\{a, c, f\}) = \chi(A \cup C) \leq 4 - \chi(A \cap C) \leq 2.$$

Hence $\{a, c, f\}$ is support of a triangle T'. Since $0 = T_a \cdot T' = T_a(a) \cdot T'(a)$, $T'(a) = T_a(a) = y$. Similarly, we have $T'(c) = y$. Hence, we may set $T_1 = T_2 = T'$. ∎

4. Minimally 3-connected isotropic systems

We will say that an isotropic system (L,V) is *minimally 3-connected* if and only if it is 3-connected and, at any $v\in V$, at least two minors are not 3-connected.

Lemma 4.1. *Let $S=(L,V)$ be an isotropic system such that:*
(i) $|V|>4$,
(ii) *there exist neither singular elements nor twins,*
(iii) *there exists a 3-connected minor.*
Then S is 3 connected.

Proof. Suppose that S satisfies the hypothesis but is not 3-connected. Let $L'=L|_x^v$ be a 3-connected minor. Suppose that A is a 2-separation of L and let χ and χ' be the connectivity functions of L and L' respectively. By working with $V\backslash A$ instead of A, one may assume without loss of generality that $|(V\backslash v)\backslash A|\geq 2$. Thus $\chi(A)\leq 1$, $\chi'(A\backslash v)\leq 1$, and so $|A\backslash v|\leq 1$. Therefore A is a couple of twins in L, a contradiction. ∎

Lemma 4.2. *Any cyclic isotropic system is minimally 3-connected.*

Proof. Suppose that C_n $(n>4)$ is a fundamental graph of the isotropic system $S=(L,V)$ and pick $v\in V$. As there exist two triangles T and T' such hat $0\neq T(v)\neq T'(v)\neq 0$, two of the three minors at v contain twins and are obviously not 3-connected. We have to prove that S is 3-connected: we will proceed by induction.

The conclusion is true for $n=5$, by Corollary 3.4.

Suppose $n>5$. Obviously, S contains neither singular elements nor twins. Pick $v\in V$. As mentioned in Section 2.5, there exists a minor on v which is cyclic. This minor is 3-connected by induction. Hence, S is 3-connected by Lemma 4.1. ∎

Theorem 4.3. *Let $S=(L,V)$ be a minimally 3-connected isotropic system, with $n=|V|>4$. Then S is cyclic. Furthermore, if for one element of V no minor is 3-connected then $n=5$.*

Proof.

Claim 1. *For any $c\in V$, if $S|_x^c$ and $S|_y^c$ are not 3-connected then there exist two triangles T and T' such that $T(c)=x$ and $T'(c)=y$.*

Lemma 3.5 implies that one of these two triangles — say T — must exist. Pick $b\in\sigma(T)$, $b\neq c$. By Lemma 3.6 applied to b and to the triangle T, there exists a triangle T' such that $T'(c)=y$.

Pick $c\in V$. By interchanging x, y and z if necessary, we may assume that $S|_x^c$ and $S|_y^c$ are not 3-connected. By Claim 1, we may assume that there exist two triangles T_c and T_b such that $T_c(c)=y$ and $T_b(c)=x$. By Lemma 3.1, there exists $b\in V$, $b\neq c$, such that $0\neq T_b(b)\neq T_c(b)\neq 0$. We may assume that $T_b(b)=y$ and $T_c(b)=x$. Set $\sigma(T_b)=\{a,b,c\}$ and $\sigma(T_c)=\{b,c,d\}$.

Claim 2. *If no minor of S at c is 3-connected then C_5 is a fundamental graph of S.*

By Corollary 3.4, it is sufficient to prove that $n=5$. Assume that $T_c(d)=x$ and that $S|_y^d$ is not 3-connected. By Lemma 3.4 applied to b and T_c, there exist two triangles T'_c and T_d such that $T_d(d)=y$, $T'_c(c)=z$, $T'_c(b)=T_d(b)$ (for $S|_z^c$ is not 3-connected).

Case 1. $T_d(b)\neq 0$.
Then $T'_c(b) = T_d(b) = z$ for otherwise $\langle T'_c,T_b\rangle \neq 0$ or $\langle T'_c,T_b\rangle \neq 0$. As $T_d(c) = 0$, $\langle T_b,T_d\rangle = \langle x,T_d(a)\rangle + \langle y,z\rangle$. Then, $x\neq T_d(a)\neq 0$ for $\langle T_b,T_b\rangle = 0$. Thus T_b, T_c and T_d are three independent vectors. Since the supports are included in $\{a,b,c,d\}$, we have $\chi(\{a,b,c,d\})\leq 1$, where χ is the connectivity function of S. Therefore, since S is 3-connected, this implies that $|V\backslash\{a,b,c,d\}|\leq 1$.

Case 2. $T_d(b)=0$.
Then $T'_c(d)\neq 0$, and we conclude as in Case 1, with triangles T_b, T_c and T'_c.

From now on, we may suppose that, if an element belongs to three triangles, the values of these triangles at v are not pairwise distinct (otherwise none of the three minors at v is 3-connected).

Claim 3. *If an element belongs to four triangles, then C_5 or C_6 is a fundamental graph of S.*

Suppose that c belongs to four distinct triangles T_1, T_2, T_3 and T_4. We may assume, without loss of generality, that $T_1=T_b$ and $T_2=T_c$. By the assumption above, $T_b(c)$, $T_c(c)$ and $T_3(c)$ are not pairwise distinct. Set $T_3(c)=T_b(c)=x$. By Lemma 3.3 applied to T_3 and T_b and to T_3 and T_c, $\sigma(T_3)=\{c,d,e\}$, where e is a new element of V. Similarly, $T_b(c)$, $T_c(c)$ and $T_3(c)$ are not pairwise distinct. If $T_4(c)=T_b(c)$, then $\sigma(T_4)=\{c,d,e\}=\sigma(T_3)$, which is impossible by Lemma 3.3. Therefore, $T_4(c)=T_c(c)$. Lemma 3.3 implies the following equalities:
— $\sigma(T_c)\cap\sigma(T_4)=\{c\}$,
— $|\sigma(T_b)\cap\sigma(T_4)|=|\sigma(T_d)\cap\sigma(T_4)|=2$,
— $\sigma(T_4)=\{a,c,e\}$.
Furthermore, $T_4(a)\neq T_b(a)$ and $T_4(e)\neq T_3(e)$. Hence the four vectors T_b, T_c, T_3 and T_4 are independent. Then $\chi(\{a,b,c,d,e\})\leq 1$ and $|V\backslash\{a,b,c,d,e\}|\leq 1$.

If $V=\{a,b,c,d,e\}$ then C_5 is a fundamental graph of S. Suppose that $|V|=6$ and set $V=\{a,b,c,d,e,f\}$. Then

$$\chi(\{d,e,f\} = \chi(V\backslash\{d,e,f\}) = 2.$$

Hence $\{d,e,f\}$ is the support of a triangle. Similarly, $\{e,f,a\}$ and $\{f,a,b\}$ are supports of triangles. One easily verifies that C_6 is a fundamental graph of S, as shown by the array below:

	T_b	T_c	T_d	T_e	T_f	T_a
a	x	0	0	0	x	y
b	y	x	0	0	0	x
c	x	y	x	0	0	0
d	0	x	y	x	0	0
e	0	0	x	y	x	0
f	0	0	0	x	y	x

From now on, we assume that no element belongs to four distinct triangles. By applying Lemma 3.4 to c and T_c (as defined in Claim 2), we get a third triangle T_d. By interchanging b and c if necessary, we may assume that $T_d(b)=0$ and $T_c(c)\neq$

$T_d(c) \neq 0$. Since the three values $T_b(c)$, $T_c(c)$ and $T_d(c)$ cannot be pairwise distinct, we have $T_d(c) = T_b(c) = x$. Hence, by Lemma 3.3, $T_d(a) = 0$. Let e be the third element of T_d. We may suppose $T_d(d) = y$ and $T_d(e) = x$. Let us rename a, b, c, d, e by s_1, s_2, s_3, s_4, s_5 and the triangles T_b, T_c, T_d, T'_e by T_2, T_3, T_4, T_5. Thus, for $2 \leq i \leq 4$, $T_i(s_i) = y$, $T_i(s_{i-1}) = T_i(s_{i+1}) = x$.

By Lemma 3.6 applied to s_{i+1} and T_i, we can continue this construction until we get a triangle T_n with $T_n(s_n) = y$, $T_n(s_{n-1}) = x$, $T_n(s_k) = x$ with $k < n-1$. We may suppose $n \geq 5$.

Claim 4. $V = \{s_1, s_2, \ldots, s_n\}$.

Suppose that $k \neq 1$. Then s_k belongs to T_{k-1}, T_k, T_{n-1}, T_n, which is impossible by the preceding assumption. So, $k = 1$. Therefore, by Lemma 3.3, $T_n(s_k) = T_n(s_1) = T_2(s_1) = x$. By Lemma 3.4 applied to s_1 and T_n, we get a triangle T_1 such that $0 \neq T_1(s_1) \neq T_2(s_1)$. Clearly, by Lemma 3.3, $\sigma(T_1) = \{s_n, s_1, s_2\}$. Let $V' = \{s_1, s_2, \ldots, s_n\}$. One can remark that $\{T_1, T_2, \ldots T_n\}$ is an independent set of $L \times V'$. Therefore, since S is connected, $V = V'$.

One can easily verify that C_n is a fundamental graph of S, as shown by the array below:

	T_2	T_3	T_4	T_5	$\ldots$	T_n	T_1
s_1	x	0	0	0	$\ldots$	x	y
s_2	y	x	0	0	$\ldots$	0	x
s_3	x	y	x	0	$\ldots$	0	0
s_4	0	x	y	x	$\ldots$	0	0
$\ldots$							
s_{n-1}	0	0	0	0	$\ldots$	x	0
s_n	0	0	0	0	$\ldots$	y	x

∎

5. Applications

Proposition 5.1. *Let $S = (L, V)$ be an isotropic system. If S is bipartite then there exist two supplementary vectors X and Y of K^V such that*

$$l = \hat{X} \cap L + \hat{Y} \cap L.$$

Proof. Assume that S is derived from a binary matroid M. Let $\mathscr{C}$ (resp. $\mathscr{C}^*$) be the linear subspace of 2^V spanned by the circuits of M (resp. the cocircuits of M). Then, there exist two supplementary vectors X and Y of K^V such that $L = \{X \cdot C + Y \cdot C' : C \in \mathscr{C}, C' \in \mathscr{C}^*\}$. This implies that $\hat{X} \cap L = \{X \cdot C : C \in \mathscr{C}\}$ (resp. $\hat{Y} \cap L = \{Y \cdot C' : C' \in \mathscr{C}^*\}$). Then, it follows that $L = \hat{X} \cap L + \hat{Y} \cap L$. ∎

Remark. The converse is true, as proved in [1].

As defined in [9], a matroid M on a set V is said to be *minimally 3-connected* if M is 3-connected and, for any $v \in V$, each of the two minors M/v and $M \setminus v$ is not 3-connected. Let W_n be a wheel, for some integer $n \geq 3$, and let $M = \mathcal{M}(W_n)$ be the cycle matroid of W_n. One easily verifies that M is minimally 3-connected. Denote by B the set of the spokes. Clearly, B is a base of M. Therefore, the fundamental graph associated with B is C_{2n}. A theorem of Tutte says that any minimally 3-connected binary matroid is isomorphic to $\mathcal{M}(W_n)$ for some integer n.

Suppose that M is minimally 3-connected binary matroid and let S be the isotropic system derived from M. Then at least two minors of S at v are not 3-connected. Hence S is minimally 3-connected and, by Theorem 4.3, is cyclic. Conversely, Theorem 4.3 implies the binary part of Tutte's theorem. We have just to prove:

Lemma 5.2. *Let $S = (L, V)$ be a cyclic isotropic system. Then S is bipartite if and only if $|V|$ is even.*

Proof.

(i) Suppose that n is even. Then the graph C_n is bipartite and so S is bipartite.

(ii) Suppose that S is bipartite. By Proposition 5.1, there exist two supplementary vectors X and Y such that $L = L \cap \hat{X} + L \cap \hat{Y}$. Let T be a triangle of L. Then $T = T_X + T_{Y'}$, with $T_X \in L \cap \hat{X}$ and $T_Y \in L \cap \hat{Y}$. By Lemma 4.2, S is 3-connected. Hence, each of the vectors T_X and T_Y is either the null vector or a triangle. Moreover, $T = T_X$ or $T = T_Y$. Let $\mathcal{I}$ the set of triangles of L. We have proved that $\{\mathcal{I} \cap \hat{X}, \mathcal{I} \cap \hat{Y}\}$ is a partition of $\mathcal{I}$. Define a simple graph $\Gamma = (\mathcal{I}, \mathcal{E})$ as following:

$TT' \in \mathcal{E}$ if and only if there exists $v \in V$ such that $0 \neq T(v) \neq T'(v) \neq 0$.

Let T_1 and T_2 be two triangles of S. Clearly, if T_1 and T_2 belong to the same class of the bipartition $\{\mathcal{I} \cap \hat{X}, \mathcal{I} \cap \hat{Y}\}$, T_1 and T_2 cannot be adjacent in Γ. Hence Γ is bipartite. But, if C_n is a fundamental graph of S, C_n is isomorphic to a subgraph of Γ (actually, C_n is isomorphic to Γ for $n \geq 7$). This implies that n is even. ∎

6. Prime graphs

Suppose that $G = (V, E)$ is a simple graph. A *split*, as defined in [7] is a partition $\{A, B\}$ of E such that:

— $|A| \geq 2$, $|B| \geq 2$,

— the cocircuit $\delta(A, B)$ (i.e. the set $\{ab \in E : a \in A, b \in B\}$) is the set of the edges of a complete bipartite graph, with chromatic classes $A' \subseteq A$ and $B' \subseteq B$.

G is said to be *prime* is there exist no splits. As proved in [6], a graph with at least four vertices is prime if and only if the isotropic system S having G as fundamental graph is 3-connected. Indeed $\{A, B\}$ is a split of G if and only if A and B are 2-separations of S.

Suppose that $G = (V, E)$ is a prime graph with at least six vertices and that $S = (L, V)$ is the isotropic system having G as fundamental graph. Then G is not locally equivalent to C_5 and so, by Theorem 4.3, there exists a 3-connected minor $S|_t^v$ of S, where $v \in V$. Then, as mentioned in Section 2.4, one out of the graphs $G \backslash v$, $G^*v \backslash v$ and $(G^*vwv)/v$ (where w is a neighbour of v) is prime. This result has been proved directly in [2]. We give a strengthening of this result:

Theorem 5.3. *Let $G=(V,E)$ be a prime graph, with $|V|\geq 6$. There exists a vertex v such that $G\backslash v$ or $G^*v\backslash v$ is prime.*

Proof. Suppose that (G,A,B) is a graphic presentation of an isotropic system $S=(L,V)$ and set $n=|V|$. As mentioned in Section 2, $G\backslash v$ (resp. $G^*v\backslash v$) is a fundamental graph of $S|^v_{A(v)}$ (resp. $S|^v_{A(v)+B(v)}$).

(i) Suppose that S is not cyclic. By Theorem 4.3, there exists $v\in V$ such that two minors at v are 3-connected. One of these minors is not equal to $S|^v_{B(v)}$. Hence, $G\backslash v$ or $G^*v\backslash v$ is a fundamental graph of this 3-connected minor and so it is prime.

(ii) Suppose that S is cyclic. Let $(C_n,\ X,\ Y)$ be a graphic presentation of S having C_n as fundamental graph. We may assume without loss of generality that $X(v)=x$ and $Y(v)=y$ for any vertex v. Suppose that the sequence of vertices in C_n is $(s_1,s_2,\ldots,s_n)$. Let $(T_i:1\leq i\leq n)$ be the fundamental basis associated with this presentation. Then, for $1\leq i\leq n$ (the indices are computed modulo n), we have:

— $T_i(s_i)=y$,

— $T_i(s_{i+1})=X(s_{i-1})=x,\quad T_i(s_{i-1})=X(s_{i-1})=x$,

— $T_i(s_j)=0$ if $i\notin\{i-1,i,i+1\}$.

Hence, for any $v\in V$, none of the minors $S|^v_x$ and $S|^v_y$ is 3-connected, for each of them contains twins. Then the single 3-connected minor at v is $S|^v_z$. We have to prove $B\neq X+Y$ or, equivalently, that there exists v such that $B(v)\neq z$.

Suppose, on the contrary, that $B=X+Y$. Let $(U_i:1\leq i\leq n)$ be the fundamental basis associated to the graphic presentation (G,A,B). Then $U(v_i)=B(v_i)=z$ for $1\leq i\leq n$. Since $U_1\cdot T_1=0$ and $U_1(s_1)\cdot T_1(s_1)\neq 0$, we have $U_1(s_n)\cdot T_1(s_n)\neq 0$ or $U_1(s_2)\cdot T_1(s_2)\neq 0$. By symmetry, we may assume that $U_1(s_2)\cdot T_1(s_2)\neq 0$. Hence $0\neq U_1(s_2)=A(s_2)\neq B(s_2)$, $U_1(s_2)\neq T_1(s_2)=x$. Therefore, $U_1(s_2)=y$. Let k be an integer such that, for $1\leq i\leq k$, $U_1(s_i)=y$. If $k\neq n$, the same argument applied to T_k instead of T_1, implies that $U_1(s_{k+1})=y$. Hence, $U_1(v)=y$ for any $v\in V$. But then $U_1\cdot T_1=U_1(s_n)\cdot T_1(s_n)+U_1(s_1)\cdot T_1(s_1)+U_1(s_2)\cdot T_1(s_2)=y\cdot x+z\cdot y+y\cdot x\neq 0$, a contradiction for U_1 and T_1 belong to L. ∎

References

[1] L. ALLYS: 3-connexité dans les systèmes isotropes, Thèse, Université du Maine (1992).

[2] A. BOUCHET: Reducing prime graphs and recognizing circle graphs, *Combinatorica* **7** (1987), 243–254.

[3] A. BOUCHET: Isotropic systems, *European J. of Combin.* **8** (1987), 231–244.

[4] A. BOUCHET: Digraph decomposition and Eulerian systems, *SIAM J. Alg. Disc. Meth.* **8** (1987), 323–337.

[5] A. BOUCHET: Graphic representations of isotropic systems, *J. Combinatorial Theory B* **45** (1988), 58–76.

[6] A. BOUCHET: Connectivity of isotropic systems, *Annals of the New York Academy of Sciences* **555** (1989), 81–93.

[7] W. CUNNINGHAM and J. EDMONDS: A combinatorial decomposition theory, *Canad. J. Math. XXXII* (1982), 734–765.

[8] W. CUNNINGHAM: Decomposition of directed graphs, *SIAM J. Alg. Disc. Meth.* **3** (1987), 214–218.

[9] W. TUTTE: Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.
[10] D. J. A. WELSH: Matroid Theory, *London Math. Soc. Monographs* **8**, Academic Press, New York, 1976.

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